

Horizon area-angular momentum-charge-magnetic fluxes inequalities in 5D Einstein-Maxwell-dilaton gravity

Stoytcho Yazadjiev*

*Department of Theoretical Physics, Faculty of Physics, Sofia University
5 J. Bourchier Blvd., Sofia 1164, Bulgaria*

Abstract

In the present paper we consider 5D spacetimes satisfying the Einstein-Maxwell-dilaton gravity equations which are $U(1)^2$ axisymmetric but otherwise highly dynamical. We derive inequalities between the area, the angular momenta, the electric charge and the magnetic fluxes for any smooth stably outer marginally trapped surface.

1 Basic notions and setting the problem

The study of inequalities between the horizon area and the other characteristics of the horizon has attracted a lot of interest recently. Within the general theory of relativity, lower bounds for the area of dynamical horizons in terms of their angular momentum or/and charge were given in [1]–[8], generalizing the similar inequalities for the stationary black holes [9]–[11]. These remarkable inequalities are based solely on general assumptions and they hold for any axisymmetric but otherwise highly dynamical horizon in general relativity. For a nice review on the subject we refer the reader to [12]. The relationship between the proofs of the area-angular-momentum-charge inequalities for quasilocal black holes and stationary black holes is discussed in [13]–[15]. Inequalities between the horizon area, the angular momentum, and the charges were also studied in some 4D alternative gravitational theories [16].

A generalization of the 4D horizon area-angular momentum inequality to D-dimensional vacuum Einstein gravity with $U(1)^{D-3}$ group of spatial isometries was given in [17]. The purpose of the present work is to derive some inequalities between the horizon area, horizon angular momentum, horizon charges and magnetic fluxes in the 5D Einstein-Maxwell-dilaton gravity including as a particular case the 5D Einstein-Maxwell gravity. It should be stressed that the derivation of the mentioned inequalities in the higher dimensional Einstein-Maxwell and Einstein-Maxwell-dilaton gravity is much more difficult and is not so straightforward as in the higher dimensional vacuum gravity even in spacetimes admitting $U(1)^{D-3}$

*yazad@phys.uni-sofia.bg

isometry group. The main reason behind this is the lack of nontrivial group of hidden symmetries for the dimensionally reduced Einstein-Maxwell-dilaton equations in the general case ¹[18]. In contrast, the dimensionally reduced vacuum Einstein equations (in space-times with $U(1)^{D-3}$ isometry group) possess nontrivial group of hidden symmetries, namely $SL(D-2, \mathbb{R})$ and a matrix sigma model presentation is possible. Some of the difficulties due to the presence of a Maxwell field can be circumvented by following a method similar to that used in the 4D Einstein-Maxwell-dilaton gravity [16] as we show below.

Let $(\mathcal{M}, g_{ab}, F_{ab}, \varphi)$ be a 5-dimensional spacetime satisfying the Einstein-Maxwell-dilaton equations

$$G_{ab} = 2\partial_a\varphi\partial_b\varphi - \nabla^c\varphi\nabla_c\varphi g_{ab} - 2V(\varphi)g_{ab} + 2e^{-2\alpha\varphi}\left(F_{ac}F_b{}^c - \frac{g_{ab}}{4}F_{cd}F^{cd}\right), \quad (1)$$

$$\nabla_a\left(e^{-2\alpha\varphi}F^{ab}\right) = 0 = \nabla_{[a}F_{bc]}, \quad (2)$$

$$\nabla_a\nabla^a\varphi = \frac{dV(\varphi)}{d\varphi} - \frac{\alpha}{2}e^{-2\alpha\varphi}F_{cd}F^{cd}, \quad (3)$$

where g_{ab} is the spacetime metric, ∇_a is its Levi-Civita connection, $G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R$ is the Einstein tensor and F_{ab} is the Maxwell field. The dilaton field is denoted by φ , $V(\varphi)$ is its potential and α is the dilaton coupling parameter. We assume that the dilaton potential is non-negative, $V(\varphi) \geq 0$. The Einstein-Maxwell gravity is recovered by first putting $\alpha = 0$ and $V(\varphi) = 0$ and then $\varphi = 0$.

As an additional technical assumption we require the spacetime to admit $U(1)^2$ group of spatial isometries. The commuting Killing fields are denoted by η_1 and η_2 and they are normalized to have a period 2π . We also require the Maxwell and the dilaton fields to be invariant under the flow of the Killing fields, i.e. $\mathcal{L}_{\eta_I}F = \mathcal{L}_{\eta_I}\varphi = 0$.

Let us further consider a compact closed smooth submanifold \mathcal{B} of dimension $\dim \mathcal{B} = 3$ invariant under the action of $U(1)^2$. The induced metric on \mathcal{B} and its Levi-Civita connection are denoted by γ_{ab} and D_a , respectively. The future directed null normals to \mathcal{B} will be denoted by n and l with the normalization condition $g(n, l) = -1$ and with $-l$ pointing outward. In what follows we require \mathcal{B} to be a stably outer marginally trapped surface which means that $\Theta_n = 0$ and $\mathcal{L}_l\Theta_n \leq 0$ with Θ_n being the expansion of n on \mathcal{B} .

As a 3-dimensional compact manifold with an action of $U(1)^2$, \mathcal{B} is topologically either S^3 , $S^2 \times S^1$ or a lens space $L(p, q)$ with p and q being co-prime integers [22, 23]. Moreover, the factor space $\hat{\mathcal{B}} = \mathcal{B}/U(1)^2$ can be identified with the closed interval $[-1, +1]$. As it was shown in [22, 23], certain linear combinations of the Killing fields η_I , with integer coefficients, vanish at the ends of the factor space. In other words, there exist integer vectors $\mathbf{a}_{\pm} \in \mathbb{Z}^2$ such that $a_{\pm}^I \eta_I \rightarrow 0$ at $x = \pm 1$, where x is the coordinate parameterizing the factor space. Equivalently, the Gram matrix defined by

$$H_{IJ} = g(\eta_I, \eta_J) \quad (4)$$

is invertible in the interior of the interval $[-1, 1]$ and has one-dimensional kernel at the interval end points, i.e. $H_{IJ}a_{\pm}^I \rightarrow 0$ at $x = \pm 1$.

In fact, the integer vectors \mathbf{a}_{\pm} determine the topology of \mathcal{B} . By a global $SL(2, \mathbb{Z})$ redefinition of the Killing fields [22, 23] we may present \mathbf{a}_{\pm} in the form $\mathbf{a}_+ = (1, 0)$ and $\mathbf{a}_- = (p, q)$

¹Fortunately, there are sectors in Einstein-Maxwell-gravity which are completely integrable [19]-[21].

with p and q being coprime integers. The topology of \mathcal{B} is then S^3 when $(p = \pm 1, q = 0)$, $S^2 \times S^1$ when $(p = 0, q = \pm 1)$ and that of a lens space $L(p, q)$ in the other cases.

Proceeding further we consider a small neighborhood O of \mathcal{B} . When the neighborhood is sufficiently small it can be foliated by two-parametric copies $\mathcal{B}(u, r)$ of $\mathcal{B} = \mathcal{B}(0, 0)$ and parameterized by the so-called null Gauss coordinates defined by a well-known procedure [24]. In Gauss null coordinates the metric in O can be written in the form

$$g = -2du(dr - r^2\Upsilon du - r\beta_a dy^a) + \gamma_{ab} dy^a dy^b, \quad (5)$$

where $n = \frac{\partial}{\partial u}$, $l = \frac{\partial}{\partial r}$, and the function Υ and the metric γ are invariantly defined on each $\mathcal{B}(u, r)$. Using these coordinates one can show that on \mathcal{B} it holds

$$R_\gamma - D^a \beta_a - \frac{1}{2} \beta^a \beta_a - 2G_{ab} n^a l^b = -2\mathcal{E}_l \Theta_n \geq 0, \quad (6)$$

where it has been taken into account that $\Theta_n = 0$ on \mathcal{B} . Here R_γ and D^a are the Ricci scalar curvature and Levi-Civita connection with respect to the metric γ_{ab} on \mathcal{B} . Taking into account that the dilaton potential is nonnegative this inequality can be rewritten in the form

$$R_\gamma - D^a \beta_a - \frac{1}{2} \beta^a \beta_a - 2\tilde{G}_{ab} n^a l^b = 2V(\varphi) - 2\mathcal{E}_l \Theta_n \geq 0, \quad (7)$$

where $\tilde{G}_{ab} = G_{ab} + 2V(\varphi)g_{ab}$.

Making use of (7), for every axisymmetric function f (i.e. every function f invariant under the isometry group) we have

$$\begin{aligned} 0 &\leq \int_{\mathcal{B}} \left(-D^a \beta_a - \frac{1}{2} \beta^a \beta_a + R_\gamma - 2\tilde{G}_{ab} n^a l^b \right) f^2 dS \\ &= \int_{\mathcal{B}} \left(2f\beta^a D_a f - \frac{1}{2} \beta^a \beta_a f^2 + R_\gamma f^2 - 2\tilde{G}_{ab} n^a l^b f^2 \right) dS, \end{aligned} \quad (8)$$

where dS is the surface element on \mathcal{B} . Now we consider the unit tangent vector N^a on \mathcal{B} which is orthogonal to η_l . With its help and taking into account that $(\gamma^{ab} - N^a N^b) D_b f = 0$, we find $\beta_a \beta^a = (\gamma^{ab} - N^a N^b) \beta_a \beta_b + (N^a \beta_a)^2$ and $2f\beta^a D_a f = 2(fN^a \beta_a) (N^b D_b f)$ which gives

$$0 \leq \int_{\mathcal{B}} \left[2(fN^a \beta_a) (N^b D_b f) - \frac{1}{2} (N^a \beta_a)^2 f^2 - \frac{1}{2} (\gamma^{ab} - N^a N^b) \beta_a \beta_b f^2 + R_\gamma f^2 - 2\tilde{G}_{ab} n^a l^b f^2 \right] dS. \quad (9)$$

Finally, taking into account that

$$2(fN^a \beta_a) (N^b D_b f) - \frac{1}{2} (N^a \beta_a)^2 f^2 \leq 2(N^b D_b f)^2 \quad (10)$$

and that $(N^b D_b f)^2 = (\gamma^{ab} - N^a N^b) D_a f D_b f + N^a N^b D_a f D_b f = \gamma^{ab} D_a f D_b f = D_a f D^a f$ we obtain the important inequality

$$0 \leq \int_{\mathcal{B}} \left[2D_a f D^a f - \frac{1}{2} (\gamma^{ab} - N^a N^b) \beta_a \beta_b f^2 + R_\gamma f^2 - 2\tilde{G}_{ab} n^a l^b f^2 \right] dS. \quad (11)$$

In order to extract the constructive information from this inequality we should perform a dimensional reduction and express the inequality as an inequality on the factor space $\hat{\mathcal{B}} = \mathcal{B}/U^2(1) = [-1, 1]$. The dimensional reduction can be performed along the lines of [18]. So we shall give here only some basic steps and results without going into detail. As a first step it is very convenient to present the Killing fields η_I in adapted coordinates, i.e. $\eta_I = \frac{\partial}{\partial \phi^I}$ where the coordinates ϕ^I are 2π -periodic. Then the induced metric γ_{ab} on \mathcal{B} takes the form

$$\gamma = \frac{dx^2}{C^2 h} + H_{IJ} d\phi^I d\phi^J, \quad (12)$$

where $C > 0$ is a constant and $h = \det(H_{IJ})$. The absence of conical singularities requires the following condition to be satisfied

$$\lim_{x \rightarrow \pm 1} C^2 \frac{h}{1-x^2} \frac{H_{IJ} a_{\pm}^I a_{\pm}^J}{1-x^2} = 1. \quad (13)$$

The area \mathcal{A} of \mathcal{B} can be easily found from (12) and the result is

$$\mathcal{A} = 8\pi^2 C^{-1}. \quad (14)$$

Therefore the condition (13) can be rewritten in the form

$$\frac{\mathcal{A}}{8\pi^2} = \lim_{x \rightarrow 1} \left(\frac{h}{1-x^2} \frac{H_{IJ} a_{+}^I a_{+}^J}{1-x^2} \right)^{1/4} \lim_{x \rightarrow -1} \left(\frac{h}{1-x^2} \frac{H_{IJ} a_{-}^I a_{-}^J}{1-x^2} \right)^{1/4}. \quad (15)$$

Since the factor space $O/U(1)^2$ is simply connected we can introduce electromagnetic potentials Φ_I and Ψ invariant under the isometry group and defined by

$$d\Phi_I = i_{\eta_I} F, \quad d\Psi = e^{-2\alpha\phi} i_{\eta_2} i_{\eta_1} \star F. \quad (16)$$

The Maxwell 2-form can then be written in the form

$$F = H^{IJ} \eta_I \wedge d\Phi_J + h^{-1} e^{2\alpha\phi} \star (d\Psi \wedge \eta_1 \wedge \eta_2). \quad (17)$$

Using the field equations one can show that there exist potentials χ_I invariant under the isometry group such that the twist $\omega_I = \star(\eta_1 \wedge \eta_2 \wedge d\eta_I)$ satisfies

$$\omega_I = d\chi_I + 2\Phi_I d\Psi - 2\Psi d\Phi_I. \quad (18)$$

By direct computation of the twist using the metric (5) one finds that on \mathcal{B} it holds $\beta_I = i_{\eta_I} \beta = C i_{\frac{\partial}{\partial x}} \omega_I$ or in explicit form

$$\beta_I = \partial_x \chi_I + 2\Phi_I \partial_x \Psi - 2\Psi \partial_x \Phi_I. \quad (19)$$

Also one can show that on \mathcal{B} we have

$$\tilde{G}_{ab} n^a l^b = D_a \phi D^a \phi + 2e^{-2\alpha\phi} H^{IJ} D_a \Phi_I D^a \Phi_J + h^{-1} e^{2\alpha\phi} D_a \Psi D^a \Psi. \quad (20)$$

Using the explicit form (12) of the metric induced on \mathcal{B} , by direct computation we find

$$R_\gamma = C^2 h \left[-\frac{\partial_x^2 h}{h} + \frac{1}{4} h^{-2} (\partial_x h)^2 - \frac{1}{4} \text{Tr} (H^{-1} \partial_x H)^2 \right]. \quad (21)$$

Finally, choosing

$$f = \left(\frac{1-x^2}{h} \right)^{1/2}, \quad (22)$$

substituting (19), (20) and (21) into (11) and taking into account that $dS = C^{-1} dx \prod_I d\phi^I$, we obtain

$$\begin{aligned} & \int_{-1}^1 \left\{ (1-x^2) \left[\frac{1}{8} \text{Tr} (H^{-1} \partial_x H)^2 + \frac{1}{8} h^{-2} (\partial_x h)^2 + \right. \right. \\ & \frac{1}{4} h^{-1} H^{IJ} (\partial_x \chi_I + 2\Phi_I \partial_x \Psi - 2\Psi \partial_x \Phi_I) (\partial_x \chi_J + 2\Phi_J \partial_x \Psi - 2\Psi \partial_x \Phi_J) \\ & \left. \left. + e^{-2\alpha\phi} H^{IJ} \partial_x \Phi_I \partial_x \Phi_J + e^{2\alpha\phi} h^{-1} (\partial_x \Psi)^2 + (\partial_x \phi)^2 \right] - \frac{1}{1-x^2} \right\} dx \leq 0. \end{aligned} \quad (23)$$

Now we can introduce the strictly positive definite metric G_{AB} given by

$$\begin{aligned} G_{AB} dX^A dX^B &= \frac{1}{8} \text{Tr} (H^{-1} dH)^2 + \frac{1}{8} h^{-2} (dh)^2 + \\ & \frac{1}{4} h^{-1} H^{IJ} (d\chi_I + 2\Phi_I d\Psi - 2\Psi d\Phi_I) (d\chi_J + 2\Phi_J d\Psi - 2\Psi d\Phi_J) + \\ & e^{-2\alpha\phi} H^{IJ} d\Phi_I d\Phi_J + e^{2\alpha\phi} h^{-1} (d\Psi)^2 + (d\phi)^2 \end{aligned} \quad (24)$$

on the 9-dimensional manifold $\mathcal{N} = \{(H_{IJ} (I \leq J), \chi_I, \Phi_I, \Psi, \phi) \in \mathbb{R}^9; h > 0\}$. In terms of this metric the inequality (23) takes the form

$$I_*[X^A] = \int_{-1}^1 \left[(1-x^2) G_{AB} \frac{dX^A}{dx} \frac{dX^B}{dx} - \frac{1}{1-x^2} \right] dx \leq 0. \quad (25)$$

In order to transform this inequality into an inequality for the area, we use condition (15) which combined with (25) gives

$$\mathcal{A} \geq 8\pi^2 e^{I[X^A]}, \quad (26)$$

where

$$I[X^A] = I_*[X^A] + \frac{1}{4} x \ln \left[\frac{h}{1-x^2} \frac{H_{IJ} a^I(x) a^J(x)}{1-x^2} \right] \Big|_{x=-1}^{x=1} \quad (27)$$

with $a^I(x)$ defined by $a^I(x) = \frac{1}{2}(1+x)a^I_+ + \frac{1}{2}(1-x)a^I_-$. We should note that there is an ambiguity in defining the functional $I[X^A]$. For example, we can define it by

$$I[X^A] = a I_*[X^A] + \frac{1}{4} x \ln \left[\frac{h}{1-x^2} \frac{H_{IJ} a^I(x) a^J(x)}{1-x^2} \right] \Big|_{x=-1}^{x=1}, \quad (28)$$

where a is an arbitrary positive number. This ambiguity, however, does not affect the final results since $I_*[X^A] = 0$ as we show below.

2 Minimizer existence lemma

In order to put a lower bound on the area we should find the minimum of the function $I[X^A]$ with appropriate boundary conditions if the minimum exists. Below we show that in certain cases the minimum exists. The natural class of functions for the minimizing problem is given by $\sigma = -\ln\left(\frac{h}{1-x^2}\right) \in C^\infty[-1, 1]$, $\ln\left(\frac{H_{IJ} a^I a^J}{1-x^2}\right) \in C^\infty[-1, 1]$, $(\chi_I, \Phi_I, \Psi, \varphi) \in C^\infty[-1, 1]$ with boundary conditions $\sigma(\pm 1) = \sigma^\pm$, $(\chi_I(\pm 1), \Phi_I(\pm 1), \Psi(\pm 1), \varphi(\pm 1)) = (\chi_I^\pm, \Phi_I^\pm, \Psi^\pm, \varphi^\pm)$. Since the electromagnetic potentials and the twist potential are defined up to a constant, without loss of generality we can choose

$$\chi_I^+ = -\chi_I^-, \Phi_I^+ = -\Phi_I^-, \Psi^+ = -\Psi^-. \quad (29)$$

Lemma 1. *For dilaton coupling parameter satisfying $0 \leq \gamma^2 \leq \frac{8}{3}$, there exists a unique smooth minimizer of the functional $I[X^A]$ with the prescribed boundary conditions.*

Proof. Let us consider the truncated functional

$$I_*[X^A][x_2, x_1] = \int_{x_1}^{x_2} \left[(1-x^2) G_{AB} \frac{dX^A}{dx} \frac{dX^B}{dx} - \frac{1}{1-x^2} \right] dx \quad (30)$$

with $-1 < x_1 < x_2 < 1$. By introducing a new variable $t = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right)$ the truncated functional takes the form

$$I_*[X^A][t_2, t_1] = \int_{t_1}^{t_2} \left[G_{AB} \frac{dX^A}{dt} \frac{dX^B}{dt} - 1 \right] dt, \quad (31)$$

which is just a modified version of the geodesic functional in the Riemannian space (\mathcal{N}, G_{AB}) . Consequently the critical points of the functional are geodesics in \mathcal{N} . It was shown in [18] that for $0 \leq \alpha^2 \leq \frac{8}{3}$ the Riemannian space (\mathcal{N}, G_{AB}) is simply connected, geodesically complete and with negative sectional curvature. Therefore, for fixed points $X^A(t_1)$ and $X^A(t_2)$ there exist a unique minimizing geodesic connecting these points. Therefore the global minimizer of $I_*[X^A][t_2, t_1]$ exists and is unique for $0 \leq \alpha^2 \leq \frac{8}{3}$. Since (\mathcal{N}, G_{AB}) is geodesically complete the global minimizer of $I_*[X^A][t_2, t_1]$ can be extended to a global minimizer of $I_*[X^A]$. Indeed, let us take $x_1(\epsilon) = -1 + \epsilon$ and $x_2(\epsilon) = 1 - \epsilon$ (i.e. $t_1(\epsilon) = -t_2(\epsilon) = \frac{1}{2} \ln \left(\frac{\epsilon}{2-\epsilon} \right)$) with ϵ being a small positive number and consider the truncated functional

$$I_*^\epsilon[X^A] = \int_{x_1(\epsilon)}^{x_2(\epsilon)} \left[(1-x^2) G_{AB} \frac{dX^A}{dx} \frac{dX^B}{dx} - \frac{1}{1-x^2} \right] dx \quad (32)$$

with boundary conditions $X^A(x_1(\epsilon))$ and $X^A(x_2(\epsilon))$. Consider now the unique minimizing geodesic Γ_ϵ in \mathcal{N} between the points $X^A(x_1(\epsilon))$ and $X^A(x_2(\epsilon))$. Then we have

$$I_*^\epsilon[X^A] \geq I_*^\epsilon[X^A]|_{\Gamma_\epsilon} \quad (33)$$

where the right hand side of the above inequality is evaluated on the geodesic Γ_ϵ . Taking into account that $\lambda_\epsilon^2 = G_{AB} \frac{dX^A}{dt} \frac{dX^B}{dt}$ is constant on the geodesic Γ_ϵ , we obtain

$$I_*^\epsilon[X^A]|_{\Gamma_\epsilon} = \int_{t_1(\epsilon)}^{t_2(\epsilon)} \left[G_{AB} \frac{dX^A}{dt} \frac{dX^B}{dt} - 1 \right] dt = (\lambda_\epsilon^2 - 1) (t_2(\epsilon) - t_1(\epsilon)). \quad (34)$$

Our next step is to evaluate λ_ϵ^2 and this can be done by evaluating $G_{AB} \frac{dX^A}{dt} \frac{dX^B}{dt}$ at the boundary points which are in a small neighborhood of the poles $x = \pm 1$. For this purpose we first write λ_ϵ^2 in the form

$$\begin{aligned} \lambda_\epsilon^2 = & \frac{(1-x^2)^2}{8} \text{Tr} \left(H^{-1} \frac{dH}{dx} \right)^2 + \frac{(1-x^2)^2}{8} h^{-2} \left(\frac{dh}{dx} \right)^2 \\ & + \frac{(1-x^2)^2}{4} h^{-1} H^{IJ} \left(\frac{d\chi_I}{dx} + 2\Phi_I \frac{d\Psi}{dx} - 2\Psi \frac{d\Phi_I}{dx} \right) \left(\frac{d\chi_J}{dx} + 2\Phi_J \frac{d\Psi}{dx} - 2\Psi \frac{d\Phi_J}{dx} \right) \\ & + (1-x^2)^2 e^{-2\alpha\phi} H^{IJ} \frac{d\Phi_I}{dx} \frac{d\Phi_J}{dx} + (1-x^2)^2 e^{2\alpha\phi} h^{-1} \left(\frac{d\Psi}{dx} \right)^2 + (1-x^2)^2 \left(\frac{d\phi}{dx} \right)^2. \end{aligned} \quad (35)$$

Within the class of functions we consider, we have

$$\frac{(1-x^2)^2}{8}h^{-2}\left(\frac{dh}{dx}\right)^2 = \frac{1}{2} + O(\varepsilon) \quad (36)$$

in a small neighborhood of the poles.

In order to estimate the term associated with H we take into account that $H^{-1}\frac{dH}{dx}$ satisfies its own characteristic equation, namely $Tr\left(H^{-1}\frac{dH}{dx}\right)^2 = h^{-2}\left(\frac{dh}{dx}\right)^2 - 2h^{-1}\det\frac{dH}{dx}$. Hence we find

$$\frac{(1-x^2)^2}{8}Tr\left(H^{-1}\frac{dH}{dx}\right)^2 = \frac{1}{2} + O(\varepsilon). \quad (37)$$

Proceeding further we notice that $\partial/\partial\chi_I$ are Killing fields for the metric G_{AB} and consequently we have the following constants of motion on the geodesics Γ_ε

$$\frac{1}{2}h^{-1}H^{IJ}\left(\frac{d\chi_I}{dt} + 2\Phi_I\frac{d\Psi}{dt} - 2\Psi\frac{d\Phi_I}{dt}\right) = \frac{1-x^2}{2}h^{-1}H^{IJ}\left(\frac{d\chi_I}{dx} + 2\Phi_I\frac{d\Psi}{dx} - 2\Psi\frac{d\Phi_I}{dx}\right) = c_\varepsilon^I. \quad (38)$$

Hence we obtain

$$\begin{aligned} & \frac{(1-x^2)^2}{4}h^{-1}H^{IJ}\left(\frac{d\chi_I}{dx} + 2\Phi_I\frac{d\Psi}{dx} - 2\Psi\frac{d\Phi_I}{dx}\right)\left(\frac{d\chi_J}{dx} + 2\Phi_J\frac{d\Psi}{dx} - 2\Psi\frac{d\Phi_J}{dx}\right) = \\ & hH_{IJ}c_\varepsilon^Ic_\varepsilon^J = O(\varepsilon). \end{aligned} \quad (39)$$

For the remaining terms, it is easy to see that they behave as

$$(1-x^2)^2e^{-2\alpha\varphi}H^{IJ}\frac{d\Phi_I}{dx}\frac{d\Phi_J}{dx} = O(\varepsilon), \quad (40)$$

$$(1-x^2)^2e^{2\alpha\varphi}h^{-1}\left(\frac{d\Psi}{dx}\right)^2 = O(\varepsilon), \quad (41)$$

$$(1-x^2)^2\left(\frac{d\varphi}{dx}\right)^2 = O(\varepsilon^2). \quad (42)$$

Summarizing the results so far, we conclude that the behavior of λ_ε^2 for small ε is

$$\lambda_\varepsilon^2 = 1 + O(\varepsilon). \quad (43)$$

Therefore we have

$$\lim_{\varepsilon \rightarrow 0} I_*^\varepsilon[X^A]|_{\Gamma_\varepsilon} = 0 \quad (44)$$

which, in view of (33), gives

$$I_*[X^A] = \lim_{\varepsilon \rightarrow 0} I_*^\varepsilon[X^A] \geq 0. \quad (45)$$

Therefore, there exists a unique global minimizer of the functional $I_*[X^A]$. Since the functionals $I[X^A]$ and $I_*[X^A]$ differ in boundary terms the global minimizer of $I_*[X^A]$ is also a global minimizer of $I[X^A]$. This completes the proof.

It should be noted that from (25) and (45) immediately follows that $I_*[X^A] = 0$.

The extremal stationary near horizon geometry is in fact defined by the same variational problem with the same boundary conditions and by the same class of functions. Therefore, as an direct consequence of the proven lemma we obtain the following

Corollary. *For every dilaton coupling parameter α in the range $0 \leq \alpha^2 \leq \frac{8}{3}$ the area \mathcal{A} of \mathcal{B} satisfies the inequality*

$$\mathcal{A} \geq \mathcal{A}_{ENHG}, \quad (46)$$

where \mathcal{A}_{ENHG} is the area associated with the extremal stationary near horizon geometry of Einstein-Maxwell-dilaton gravity with $V(\phi) = 0$, for the corresponding α . The equality is saturated only for the area associated with extremal stationary near horizon geometry with $V(\phi) = 0$.

3 Horizon area-angular momenta-charge-magnetic fluxes inequality for critical dilaton coupling parameter

For the critical coupling $\alpha^2 = \frac{8}{3}$ the Riemannian space (\mathcal{N}, G_{AB}) is an $SL(4, \mathbb{R})/O(4)$ symmetric space [16] and therefore, there exists a matrix M such that the metric G_{AB} can be written in the form

$$G_{AB}dX^A dX^B = \frac{1}{8} \text{Tr}(M^{-1}dM)^2, \quad (47)$$

where M is positive definite and $M \in SL(4, \mathbb{R})$. Finding the explicit form of the matrix M is a tedious task and here we present only the final result. The matrix M is given by

$$M = \begin{pmatrix} E_{2 \times 2} & 0 \\ S^T & E_{2 \times 2} \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & Y \end{pmatrix} \begin{pmatrix} E_{2 \times 2} & S \\ 0 & E_{2 \times 2} \end{pmatrix} = \begin{pmatrix} N & NS \\ S^T N & S^T NS + Y \end{pmatrix}, \quad (48)$$

where $E_{2 \times 2}$ is the unit 2×2 matrix and S , N and Y are 2×2 matrices which have the following explicit form:

$$S = \begin{pmatrix} 2\Phi_1 & 2\Phi_2 \\ \chi_1 + 2\Phi_1\Psi & \chi_2 + 2\Phi_2\Psi \end{pmatrix}, \quad (49)$$

$$N = e^{\sqrt{\frac{2}{3}}\Phi} h^{-1} \begin{pmatrix} e^{-4\sqrt{\frac{2}{3}}\Phi} h + 4\Psi^2 & -2\Psi \\ -2\Psi & 1 \end{pmatrix}, \quad (50)$$

$$Y = e^{\sqrt{\frac{2}{3}}\Phi} H. \quad (51)$$

In terms of the matrix M , the Euler-Lagrange equations are

$$\frac{d}{dx} \left[(1-x^2) M^{-1} \frac{dM}{dx} \right] = 0. \quad (52)$$

Hence we obtain

$$(1-x^2) M^{-1} \frac{dM}{dx} = 2A, \quad (53)$$

where A is a constant matrix with $\text{Tr}A = 0$, since $\det M = 1$. Integrating further we find

$$M = M_0 \exp \left(\ln \frac{1+x}{1-x} A \right) \quad (54)$$

with M_0 being a constant matrix with the same properties as M and satisfying $A^T M_0 = M_0 A$. As a positive definite matrix, M_0 can be written in the form $M_0 = BB^T$ for some constant matrix B with $|\det B| = 1$ and this presentation is up to an orthogonal matrix O , i.e it is invariant under the transformation $B \rightarrow BO$. This freedom can be used to diagonalize the symmetric matrix $B^T A B^{T-1}$. So we can take $B^T A B^{T-1} = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and we obtain

$$M = B \begin{pmatrix} \left(\frac{1+x}{1-x}\right)^{\lambda_1} & 0 & 0 & 0 \\ 0 & \left(\frac{1+x}{1-x}\right)^{\lambda_2} & 0 & 0 \\ 0 & 0 & \left(\frac{1+x}{1-x}\right)^{\lambda_3} & 0 \\ 0 & 0 & 0 & \left(\frac{1+x}{1-x}\right)^{\lambda_4} \end{pmatrix} B^T. \quad (55)$$

The eigenvalues λ_i can be found by comparing the singular behavior of the left and the right hand side of (55) at $x \rightarrow \pm 1$. Taking into account that only the matrix N in M is singular at $x \rightarrow \pm 1$, we find that $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = \lambda_4 = 0$. Even more, if we write the matrix B in block form

$$B = \begin{pmatrix} B_1 & R \\ L & B_2 \end{pmatrix}, \quad (56)$$

where B_1, B_2, R and L are 2×2 matrices, from the singular behavior at $x \rightarrow \pm 1$ we find

$$B_1 E_{\pm} B_1^T = \frac{1}{4} N_{\pm}, \quad (57)$$

$$B_1 E_{\pm} L^T = \frac{1}{4} N_{\pm} S_{\pm}, \quad (58)$$

$$L E_{\pm} L^T = \frac{1}{4} S_{\pm}^T N_{\pm} S_{\pm}. \quad (59)$$

Here the matrices E_{\pm} , N_{\pm} and S_{\pm} are defined as follows

$$E_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad (60)$$

$$N_{\pm} = \lim_{x \rightarrow \pm 1} (1 - x^2) N = e^{\sqrt{\frac{2}{3}} \varphi_{\pm} + \sigma_{\pm}} \begin{pmatrix} 4\Psi^{\pm 2} & -2\Psi^{\pm} \\ -2\Psi^{\pm} & 1 \end{pmatrix}, \quad (61)$$

$$S_{\pm} = \lim_{x \rightarrow \pm 1} S = \begin{pmatrix} 2\Phi_1^{\pm} & 2\Phi_2^{\pm} \\ \chi_1^{\pm} + 2\Phi_1^{\pm} \Psi^{\pm} & \chi_2^{\pm} + 2\Phi_2^{\pm} \Psi^{\pm} \end{pmatrix}. \quad (62)$$

In order to explore the regular part of M at $x \rightarrow 1$ we consider the matrix $(1 - x)M$. Taking into account (57) we find that at $x \rightarrow 1$ we have

$$(1 - x)N = \frac{1}{2}N_+ + (1 - x)RR^T + \frac{1}{8}(1 - x)^2 N_-, \quad (63)$$

$$(1 - x)NS = \frac{1}{2}N_+ S_+ + (1 - x)RB_2^T + \frac{1}{8}(1 - x)^2 N_- S_-, \quad (64)$$

$$(1 - x)S^T NS + (1 - x)Y = \frac{1}{2}S_+^T N_+ S_+ + (1 - x)B_2 B_2^T + \frac{1}{8}(1 - x)^2 S_-^T N_- S_-. \quad (65)$$

Using these relations after long but straightforward calculations we obtain

$$\lim_{x \rightarrow 1} \frac{h}{1 - x^2} \frac{H_{IJ} a_+^I a_+^J}{1 - x^2} = \frac{e^{-\sqrt{\frac{2}{3}} \varphi_+ - \sigma_+}}{16} [s_+^T(a_+) - s_-^T(a_+)] N_- [s_+(a_+) - s_-(a_+)], \quad (66)$$

where

$$s_{\pm}(a_+) = S_{\pm} a_+ = \begin{pmatrix} 2\Phi_1^{\pm} & 2\Phi_2^{\pm} \\ \chi_1^{\pm} + 2\Phi_1^{\pm} \Psi^{\pm} & \chi_2^{\pm} + 2\Phi_2^{\pm} \Psi^{\pm} \end{pmatrix} \begin{pmatrix} a_+^1 \\ a_+^2 \end{pmatrix} = \begin{pmatrix} 2\Phi_I^{\pm} a_+^I \\ \chi_I^{\pm} a_+^I + 2\Phi_I^{\pm} a_+^I \Psi^{\pm} \end{pmatrix}. \quad (67)$$

By similar considerations one can show that

$$\lim_{x \rightarrow -1} \frac{h}{1 - x^2} \frac{H_{IJ} a_-^I a_-^J}{1 - x^2} = \frac{e^{-\sqrt{\frac{2}{3}} \varphi_- - \sigma_-}}{16} [s_+^T(a_-) - s_-^T(a_-)] N_+ [s_+(a_-) - s_-(a_-)], \quad (68)$$

where

$$s_{\pm}(a_{-}) = S_{\pm}a_{-} = \begin{pmatrix} 2\Phi_1^{\pm} & 2\Phi_2^{\pm} \\ \chi_1^{\pm} + 2\Phi_1^{\pm}\Psi^{\pm} & \chi_2^{\pm} + 2\Phi_2^{\pm}\Psi^{\pm} \end{pmatrix} \begin{pmatrix} a_{-}^1 \\ a_{-}^2 \end{pmatrix} = \begin{pmatrix} 2\Phi_I^{\pm}a_{-}^I \\ \chi_I^{\pm}a_{-}^I + 2\Phi_I^{\pm}a_{-}^I\Psi^{\pm} \end{pmatrix}. \quad (69)$$

The above results combined with (26) and (27), when $I_*[X^A] = 0$ is taken into account, give the following inequality

$$\mathcal{A} \geq 8\pi^2 (Z_+ Z_-)^{1/4}, \quad (70)$$

where

$$Z_+ = \frac{1}{16} [s_+^T(a_+) - s_-^T(a_+)] \Sigma_- [s_+(a_+) - s_-(a_+)], \quad (71)$$

$$Z_- = \frac{1}{16} [s_+^T(a_-) - s_-^T(a_-)] \Sigma_+ [s_+(a_-) - s_-(a_-)], \quad (72)$$

and

$$\Sigma_{\pm} = e^{-\sqrt{\frac{2}{3}}\Phi_{\pm} - \sigma_{\pm}} N_{\pm} = \begin{pmatrix} 4\Psi^{\pm 2} & -2\Psi^{\pm} \\ -2\Psi^{\pm} & 1 \end{pmatrix}. \quad (73)$$

In order to express the inequality in more compact form we should relate the potentials values at $x = \pm 1$ with the angular momenta, with the charges and with the magnetic fluxes. The full angular momenta J_I associated with \mathcal{B} are given by

$$\begin{aligned} J_I &= \frac{\pi}{4} \int_{\hat{\mathcal{B}}} i_{\eta_2} i_{\eta_1} \star d\eta_I - \frac{\pi}{2} \int_{\hat{\mathcal{B}}} (\Phi_I d\Psi - \Psi d\Phi_I) = \\ &= \frac{\pi}{4} \int_{\hat{\mathcal{B}}} \omega_I - \frac{\pi}{2} \int_{\hat{\mathcal{B}}} (\Phi_I d\Psi - \Psi d\Phi_I) = \frac{\pi}{4} \int_{\hat{\mathcal{B}}} d\chi_I, \end{aligned} \quad (74)$$

where the first integral is the contribution of the gravitational field while the second one reflects the contribution of the electromagnetic field. The direct calculation gives the following expressions for J_I , namely

$$J_I = \frac{\pi}{4} (\chi^+ - \chi^-) = \frac{\pi}{2} \chi^+. \quad (75)$$

The electric charge is given by

$$Q = \frac{1}{2\pi^2} \int_{\mathcal{B}} e^{-2\alpha\phi} \star F = 2 (\Psi^+ - \Psi^-) = 4\Psi^+. \quad (76)$$

In this way we obtain

$$\mathcal{A} \geq 8\pi \sqrt{|J_+ + \frac{1}{8}Q\mathfrak{F}_+| |J_- - \frac{1}{8}Q\mathfrak{F}_-|}, \quad (77)$$

where

$$J_{\pm} = J_I a_{\pm}^I, \quad \mathfrak{F}_{\pm} = 2\pi (\Phi_I^+ - \Phi_I^-) a_{\pm}^I. \quad (78)$$

The quantities \mathfrak{F}_{\pm} can be interpreted as magnetic fluxes through appropriately defined 2-surfaces \mathcal{D}_{\pm} . We define \mathcal{D}_{\pm} in the following way. First we uplift the factor space interval $\hat{\mathcal{B}} = [-1, 1]$ to a curve in the spacetime manifold \mathcal{M} and then we act with the isometries generated by the Killing field $a_{\pm}^I \eta_I$. It is not difficult to see that the so constructed 2-dimensional surfaces \mathcal{D}_{\pm} have S^2 -topology for $\mathbf{a}_+ = \pm \mathbf{a}_-$ and disk topology in the other cases. The magnetic fluxes through \mathcal{D}_{\pm} are given by

$$\begin{aligned} \mathfrak{F}_{\pm} &= \int_{\mathcal{D}_{\pm}} F = 2\pi \int_{\hat{\mathcal{B}}} i_{a_{\pm}^I \eta_I} F = 2\pi a_{\pm}^I \int_{\hat{\mathcal{B}}} i_{\eta_I} F = 2\pi a_{\pm}^I \int_{\hat{\mathcal{B}}} d\Phi_I = 2\pi a_{\pm}^I \int_{-1}^1 d\Phi_I \\ &= 2\pi a_{\pm}^I (\Phi_I^+ - \Phi_I^-) \end{aligned} \quad (79)$$

and obviously coincide with the previously defined quantities \mathfrak{F}_{\pm} . In the case when the topology of \mathcal{D}_{\pm} is the spherical one, the magnetic fluxes are in fact (up to sign) the magnetic (dipole) charge associated with \mathcal{B} .

Let us summarize the results of this section in the following

Theorem 1. *Let \mathcal{B} be a smooth stably outer marginally trapped surface in a spacetime satisfying 5D Einstein-Maxwell-dilaton equations with a dilaton coupling parameter $\alpha^2 = \frac{8}{3}$ and having isometry group $U(1)^2$. If the dilaton potential is non-negative, then the area of \mathcal{B} satisfies the inequality*

$$\mathcal{A} \geq 8\pi \sqrt{|J_+ + \frac{1}{8}Q\mathfrak{F}_+| |J_- - \frac{1}{8}Q\mathfrak{F}_-|} \quad (80)$$

with $J_{\pm} = J_I a_{\pm}^I$, where J_I , Q , and \mathfrak{F}_{\pm} are the angular momenta, the electric charge and the magnetic fluxes associated with \mathcal{B} , respectively. The equality is saturated only for the extremal stationary near horizon geometry of the $\alpha^2 = \frac{8}{3}$ Einstein-Maxwell-dilaton gravity with $V(\varphi) = 0$.

4 Horizon area-angular momenta-charge-magnetic fluxes inequality for dilaton coupling parameter $0 \leq \alpha^2 \leq \frac{8}{3}$

Finding a sharp lower bound for the horizon area for arbitrary dilaton coupling parameter is very difficult since the geodesic equations for arbitrary α can not be integrated explicitly. Nevertheless an important estimate can be found for dilaton coupling parameter in the range $0 \leq \alpha^2 \leq \frac{8}{3}$. The inequality is given by the following

Theorem 2. *Let \mathcal{B} be a smooth stably outer marginally trapped surface in a spacetime satisfying 5D Einstein-Maxwell-dilaton equations with a dilaton coupling parameter $0 \leq$*

$\alpha^2 \leq \frac{8}{3}$ and having isometry group $U(1)^2$. If the dilaton potential is non-negative, then the area of \mathcal{B} satisfies the inequality

$$\mathcal{A} \geq 8\pi \sqrt{|J_+ + \frac{1}{8}Q\mathfrak{F}_+||J_- - \frac{1}{8}Q\mathfrak{F}_-|} \quad (81)$$

with $J_\pm = J_I a_\pm^I$, where J_I , Q , and \mathfrak{F}_\pm are the angular momenta, the electric charge and the magnetic fluxes associated with \mathcal{B} , respectively. The equality is saturated for the extremal stationary near horizon geometry of the $\alpha^2 = \frac{8}{3}$ Einstein-Maxwell-dilaton gravity with $V(\phi) = 0$.

Proof. The proof is a direct generalization of the one in four dimensions [16]. Let us first consider the case $0 < \alpha^2 \leq \frac{8}{3}$ and define the metric

$$\begin{aligned} \tilde{G}_{AB} dX^A dX^B &= \frac{1}{8} \text{Tr} (H^{-1} dH)^2 + \frac{1}{8} h^{-2} (dh)^2 + \\ &\frac{1}{4} h^{-1} H^{IJ} (d\chi_I + 2\Phi_I d\Psi - 2\Psi d\Phi_I) (d\chi_J + 2\Phi_J d\Psi - 2\Psi d\Phi_J) + \\ &e^{-2\alpha\phi} H^{IJ} d\Phi_I d\Phi_J + e^{2\alpha\phi} h^{-1} (d\Psi)^2 + \frac{3\alpha^2}{8} (d\phi)^2 \end{aligned} \quad (82)$$

and the associated functional

$$\tilde{I}[X^A] = \int_{-1}^1 \left[(1-x^2) \tilde{G}_{AB} \frac{dX^A}{dx} \frac{dX^B}{dx} - \frac{1}{1-x^2} \right] dx + \frac{1}{4} x \ln \left[\frac{h}{1-x^2} \frac{H_{IJ} a^I(x) a^J(x)}{1-x^2} \right] \Big|_{x=-1}^{x=1}. \quad (83)$$

It is not difficult to see that $I[X^A] \geq \tilde{I}[X^A]$ which gives

$$\mathcal{A} \geq 8\pi^2 e^{\tilde{I}[X^A]}. \quad (84)$$

However, redefining the dilaton field $\tilde{\phi} = \sqrt{\frac{3}{8}}\alpha\phi$, we see that the functional $\tilde{I}[X^A]$ reduces to the functional $I[X^A]$ for the critical coupling $\alpha^2 = \frac{8}{3}$. Therefore we can conclude that

$$\mathcal{A} \geq 8\pi \sqrt{|J_+ + \frac{1}{8}Q\mathfrak{F}_+||J_- - \frac{1}{8}Q\mathfrak{F}_-|} \quad (85)$$

for every α in the range $0 < \alpha^2 \leq \frac{8}{3}$. The continuity argument shows that the inequality also holds for the Einstein-Maxwell case $\alpha = 0$.

5 Discussion

In the present paper we derived inequalities between the area, the angular momenta, the electric charge and the magnetic fluxes for any smooth stably outer marginally trapped surface in 5D Einstein-Maxwell-dilaton gravity with dilaton coupling parameter in the range

$0 \leq \alpha^2 \leq \frac{8}{3}$. In proving the inequalities we assumed that the dilaton potential is non-negative and the spacetime is $U(1)^2$ axisymmetric but otherwise highly dynamical. *It is worth mentioning that all of our results still hold even in the presence of matter with an axially symmetric energy momentum tensor satisfying the dominant energy condition.*

Since the considerations in the present paper are entirely quasi-local, our results can be applied to stationary axisymmetric black holes in asymptotically flat and Kaluza-Klein spacetimes, as well as in spacetimes with de Sitter asymptotic.

The approach of the present paper can be easily extended to the case of 5D Einstein-Maxwell-Chern-Simons gravity with Chern-Simons coefficient λ_{CS} .

Acknowledgements: This work was partially supported by the Bulgarian National Science Fund under Grant DMU-03/6.

References

- [1] A. Acena, S. Dain and M.E. Gabach Clement, Class. Quant. Grav. **28** 105014 (2011); [arXiv:1012.2413[gr-qc]].
- [2] S. Dain and M. Reiris. Phys. Rev. Lett. **107**, 051101 (2011); [arXiv:1102.5215[gr-qc]].
- [3] M.E. Gabach Clement, [arXiv:1102.3834[gr-qc]].
- [4] J. L. Jaramillo, M. Reiris and S. Dain, Phys. Rev. **D84**, 121503 (2011); [arXiv:1106.3743[gr-qc]].
- [5] S. Dain, J. L. Jaramillo and M. Reiris, Class. Quantum Grav. **29**, 035013 (2012); [arXiv:1109.5602[gr-qc]].
- [6] M.E. Gabach Clement and J.L. Jaramillo, [arXiv:1111.6248[gr-qc]].
- [7] W. Simon. Class. Quant. Grav. **29**, 062001 (2012); [arXiv:1109.6140[gr-qc].]
- [8] M. E. Gabach Clement, J. L. Jaramillo and M. Reiris, [arXiv:1207.6761[gr-qc]].
- [9] M. Ansorg, J. Hennig and C. Cederbaum. Gen. Rel. Grav. **43**, 1205 (2011); [arXiv:1005.3128[gr-qc]].
- [10] J. Hennig, M. Ansorg and C. Cederbaum. Class. Quantum Grav. **25** 162002 (2008);[arXiv:0805.4320[gr-qc]].
- [11] J. Hennig, C. Cederbaum and M. Ansorg, Commun. Math. Phys. **293**, 449 (2010); [arXiv:0812.2811[gr-qc]].
- [12] S. Dain. Class. Quant. Grav. **29**, 073001 (2012), [arXiv:1111.3615[gr-qc]].
- [13] P. Chrusciel, M. Eckstein, L. Nguyen and S. Szybka, Class. Quant. Grav. **28**, 245017 (2011).
- [14] M. Mars, Class. Quant. Grav. **29**, 145019 (2012).

- [15] J. Jaramillo, *Class. Quant. Grav.* **29**, 177001 (2012).
- [16] S. Yazadjiev, [arXiv:1210.4684v2[gr-qc]].
- [17] S. Hollands, *Class. Quant. Grav.* **29**, 065006 (2012); [arXiv:1110.5814[gr-qc]].
- [18] S. Yazadjiev, *JHEP* **1106**, 083 (2011); [arXiv:1104.0378 [hep-th]].
- [19] S. Yazadjiev, *Phys. Rev.* **D73**, 104007 (2006); [arXiv:hep-th/0602116].
- [20] S. Yazadjiev, *JHEP* **0607**, 036 (2006); [arXiv:hep-th/0604140].
- [21] S. Yazadjiev, *Phys. Rev.* **D78**, 064032 (2008); [arXiv:0805.1600 [hep-th]].
- [22] S. Hollands and S. Yazadjiev, *Commun. Math. Phys.* **283**, 749 (2008) [arXiv:0707.2775 [gr-qc]].
- [23] S. Hollands and S. Yazadjiev, “A uniqueness theorem for stationary Kaluza-Klein black holes“, *Comm. Math. Phys.* **32**, 631 (2011); [arXiv:0812.3036[gr-qc]]
- [24] S. Hollands, A. Ishibashi and R. Wald, *Commun. Math. Phys.* **271**, 699 (2007).